Iterative approach to Maxwell equations for dielectric media of spatially varying refractive index

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We propose an iterative method to solve the Maxwell equations for a one-dimensional model system with spatially varying permittivity. We construct solutions that are iterative in the scattering order, equivalent to the number of scattering events along the forward and backward directions. A numerical implementation of this approach is also presented.

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I. INTRODUCTION

The large variety of problems in fiber optics communication, waveguide technology, photonic and semiconductor devices, and thin-film technology involves the study of the optical response of a one-dimensional (1D) medium with specifically designed refractive index [1–5]. These problems generally require the calculation of the transmission and reflection amplitudes of an incident electromagnetic field. However, there exists no general solution to the Maxwell equations for media with an arbitrary refractive index. Traditionally, the transfer matrix approach [6] is used if the medium consists of a finite number of plane parallel dielectric slabs arranged either periodically or in a disordered fashion. In this approach the optical properties of each slab or scatterer are described by a 2×2 matrix and the net reflection or transmission amplitudes are obtained through matrix multiplication. This numerical approach can also be extended to a medium with a continuous refractive index by discretizing the medium into a finite number of slabs of infinitesimal length. Other well known approaches suggested in literature are the Green function technique [7], the invariant embedding theory [8,9], and the wave splitting theory [10]. Alternatively, differential equations in terms of a suitable combination of scattering amplitudes can also be constructed [11].

In this paper we suggest an alternative approach to the problem of scattering from a 1D medium. We show that the field propagating through a medium with an arbitrary refractive index can be expressed as a sum of fields corresponding to various scattering events. Such a solution can be generated directly from the Maxwell equations when rewritten in terms of two auxiliary fields. The auxiliary fields give rise to a pair of coupled differential equations with a familiar form [8–12], and when solved in an iterative fashion, the various orders of iteration correspond to the number of times the field undergoes forward or backward scattering in the medium. Thus, apart from being physically intuitive, in regimes where higher-order scattering events are not important, the solution simplifies both analytically and numerically. Particularly for wavelengths much larger than the size of the scatterers, few scattering events are sufficient.

II. ANALYTICAL SOLUTIONS OF THE ONE-DIMENSIONAL MAXWELL EQUATIONS

The Maxwell equations for a nonmagnetic medium with position-dependent permittivity are given by

$$\vec{\nabla}\cdot\varepsilon(\vec{r})\vec{E}=0, \tag{2.1a}$$

$$\vec{\nabla}\cdot\vec{B}=0, \tag{2.1b}$$

$$\vec{\nabla}\times\vec{E}=-\partial\vec{B}/\partial t, \tag{2.1c}$$

$$\vec{\nabla}\times\vec{B}=\varepsilon(\vec{r})\mu\partial\vec{E}/\partial t. \tag{2.1d}$$

If the field is normally incident on a medium whose refractive index varies only along the x direction, these equations take the form

$$\frac{\partial B_z(x,t)}{\partial t}=-\frac{\partial E_y(x,t)}{\partial x}, \tag{2.2a}$$

$$\varepsilon(x)\frac{\partial E_y(x,t)}{\partial t}=-\frac{1}{\mu}\frac{\partial B_z(x,t)}{\partial x}, \tag{2.2b}$$

$$\frac{\partial B_y(x,t)}{\partial t}=\frac{\partial E_z(x,t)}{\partial x}, \tag{2.2c}$$

$$\varepsilon(x)\frac{\partial E_z(x,t)}{\partial t}=\frac{1}{\mu}\frac{\partial B_y(x,t)}{\partial x}, \tag{2.2d}$$

where $E_{y,z}$ and $B_{y,z}$ are the transverse field components. Any arbitrary transverse polarization can be expressed as a linear combination of s polarization (E_y,B_z) and p polarization (B_y,E_z), which are the two linearly independent polarization modes. Consider auxiliary fields of the type

$$R_s(x,t)\equiv\frac{1}{2}\{\sqrt{\varepsilon(x)}E_y(x,t)+B_z(x,t)/\sqrt{\mu}\},$$

$$L_s(x,t)\equiv\frac{1}{2}\{\sqrt{\varepsilon(x)}E_y(x,t)-B_z(x,t)/\sqrt{\mu}\}; \tag{2.3a}$$

$$R_p(x,t)\equiv\frac{1}{2}\{\sqrt{\varepsilon(x)}E_z(x,t)-B_y(x,t)/\sqrt{\mu}\},$$

$$L_p(x,t)\equiv\frac{1}{2}\{\sqrt{\varepsilon(x)}E_z(x,t)+B_y(x,t)/\sqrt{\mu}\}. \tag{2.3b}$$

The wave equations (2.2a) and (2.2b) are equivalent to a set of coupled equations for the auxiliary fields, $R_s(x,t)$ and $L_s(x,t)$

$$\left(v(x)\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)R_s(x,t)=-\frac{1}{2}\frac{dv(x)}{dx}\{R_s(x,t)+L_s(x,t)\}, \tag{2.4a}$$

$$\left(v(x)\frac{\partial}{\partial x}-\frac{\partial}{\partial t}\right)L_s(x,t)=-\frac{1}{2}\frac{dv(x)}{dx}\{R_s(x,t)+L_s(x,t)\}, \tag{2.4b}$$

where $v(x)\equiv(\mu\varepsilon(x))^{-1/2}$ is the position-dependent velocity. A similar set of equations can be obtained for $R_p(x,t)$ and $L_p(x,t)$. The form of the generator in Eqs. (2.4) suggests that $R_s(x,t)$ and $L_s(x,t)$ represent fields propagating along the positive and negative x directions, respectively. $R_s^2(x,t)$ and $L_s^2(x,t)$ correspond to the right- and left-going photon fluxes, respectively. This can be obtained from the definitions (2.3a). One can show that $R_s^2(x,t)+L_s^2(x,t)=\frac{1}{2}[\varepsilon(x)E_y^2(x,t)+(1/\mu)B_z^2(x,t)]$ is the energy density and $v(x)(R_s^2(x,t)-L_s^2(x,t))=(1/\mu)E_y(x,t)B_z(x,t)$ is the Poynting vector of the s-polarized field. Thus, $R_s^2(x,t)-L_s^2(x,t)$ corresponds to the net photon flux at position x and at time t. In the following sections, we show that for boundary value problems, Eqs. (2.4) can be solved iteratively in the frequency domain and for initial value problems in the time domain. The same procedure can be followed for the p polarized field. Thus, we omit the subscripts s and p.

III. THE ITERATIVE SOLUTIONS IN THE FREQUENCY DOMAIN

Transforming the fields into Fourier space $\left\{\begin{smallmatrix}R(x,t)\\L(x,t)\end{smallmatrix}\right\}\equiv(1/2\pi)\int_{-\infty}^{\infty}d\omega e^{-i\omega t}\left\{\begin{smallmatrix}R(x,\omega)\\L(x,\omega)\end{smallmatrix}\right\}$, the system (2.4) becomes

$$\frac{\partial}{\partial x}R(x,\omega)=\left[\frac{i\omega}{v(x)}+a(x)\right]R(x,\omega)+a(x)L(x,\omega), \tag{3.1a}$$

$$\frac{\partial}{\partial x}L(x,\omega)=\left[-\frac{i\omega}{v(x)}+a(x)\right]L(x,\omega)+a(x)R(x,\omega), \tag{3.1b}$$

where $a(x)\equiv-1/(2v(x))dv(x)/dx$. We can eliminate the diagonal couplings by introducing the fields

$$\tilde{R}(x,\omega)\equiv\exp\left[-\int_0^x dx'\left(\frac{i\omega}{v(x')}+a(x')\right)\right]R(x,\omega), \tag{3.2a}$$

$$\tilde{L}(x,\omega)\equiv\exp\left[\int_0^x dx'\left(\frac{i\omega}{v(x')}-a(x')\right)\right]L(x,\omega), \tag{3.2b}$$

which satisfy the following set of equations:

$$\frac{\partial}{\partial x}\tilde{R}(x,\omega)=a(x)Z^2(x,\omega)\tilde{L}(x,\omega), \tag{3.3a}$$

$$\frac{\partial}{\partial x}\tilde{L}(x,\omega)=a(x)Z^{2*}(x,\omega)\tilde{R}(x,\omega), \tag{3.3b}$$

where $Z(x,\omega)\equiv\exp[-i\omega\int_0^x[dx'/v(x')]]$. We can construct unique iterative solutions to this system in increasing orders of $a(x)$ for a given boundary condition. The zeroth-order solutions are governed by

$$\frac{\partial}{\partial x}\tilde{R}^{(0)}(x,\omega)=0, \tag{3.4a}$$

$$\frac{\partial}{\partial x}\tilde{L}^{(0)}(x,\omega)=0. \tag{3.4b}$$

Consider a medium present in the region $0<x<b$. If there is a source at $x=0$ which generates a right-going wave, we have $R(x=0,\omega)\equiv f(\omega)$ and $L(b,\omega)\equiv0$, i.e., there is no source generating a left-going wave at $x=b$. With this set of boundary conditions, the solutions of Eqs. (3.4) are $\tilde{R}^{(0)}(x,\omega)=f(\omega)$ and $\tilde{L}^{(0)}(x,\omega)=0$. By using the zeroth-order solutions, higher-order solutions can be constructed from Eqs. (3.3). In general, we can recursively generate the mth-order solutions from the $(m-1)$th-order solutions by solving

$$\frac{\partial}{\partial x}\tilde{R}^{(m)}(x,\omega)=a(x)Z^2(x,\omega)\tilde{L}^{(m-1)}(x,\omega), \tag{3.5a}$$

$$\frac{\partial}{\partial x}\tilde{L}^{(m)}(x,\omega)=a(x)Z^{2*}(x,\omega)\tilde{R}^{(m-1)}(x,\omega) \tag{3.5b}$$

subject to the boundary conditions $\tilde{R}^{(m)}(0,\omega)=0$ and $\tilde{L}^{(m)}(b,\omega)=0$ for $m\neq0$. This yields the recursive solutions of the type

$$\tilde{R}^{(m)}(x,\omega)=\int_0^x dx'\,a(x')Z^2(x',\omega)\tilde{L}^{(m-1)}(x',\omega),$$

$$\tilde{L}^{(m)}(x,\omega)=-\int_x^b dx'\,a(x')Z^{2*}(x',\omega)\tilde{R}^{(m-1)}(x',\omega). \tag{3.6}$$

Transforming back to our original fields using Eqs. (3.2), we get the iterative solutions for R and L;

$$R^{(m)}(x,\omega)=\frac{Z^*(x,\omega)}{\beta(x)}\int_0^x dx'\,\beta(x')a(x')Z(x',\omega)\times L^{(m-1)}(x',\omega), \tag{3.7a}$$

$$L^{(m)}(x,\omega)=-\frac{Z(x,\omega)}{\beta(x)}\int_x^b dx'\,\beta(x')a(x')Z^*(x',\omega)\times R^{(m-1)}(x',\omega), \tag{3.7b}$$

where $\beta(x)\equiv\exp[-\int_0^x a(x')dx']=\sqrt{v(x)/v(0)}$ is the ratio of the velocities at x and $x=0$. For our boundary conditions $R(0,\omega)=\tilde{R}(0,\omega)=f(\omega)$ and $L(b,\omega)=\tilde{L}(b,\omega)=0$, we have $R^{(0)}(x,\omega)=Z^*(x,\omega)f(\omega)/\beta(x)$ and $L^{(0)}(x,\omega)=0$. In physical terms, $R^{(0)}(x,\omega)$ represents the right-going wave that does not scatter inside the medium, and thus $L^{(0)}(x,\omega)$ is zero. Since $L^{(0)}(x,\omega)=0$, using Eq. (3.7a) we obtain $R^{(1)}(x,\omega)=0$, in agreement with the fact that there cannot be any transmitted light with an odd number of scattering events for any finite medium. Using these expression for $R^{(0)}$ and $L^{(0)}$ and solving for various orders in Eqs. (3.7) yield $L^{(m)}(x,\omega)=0$ and

$$\begin{aligned}R^{(m)}(x,\omega)=(-1)^{m/2}R^{(0)}(x,\omega)\int_0^x dx_m a(x_m)Z^2(x_m,\omega)\\ \times\int_{x_m}^b dx_{m-1}a(x_{m-1})Z^{2*}(x_{m-1},\omega)\times\cdots\\ \times\int_0^{x_3} dx_2 a(x_2)Z^2(x_2,\omega)\\ \times\int_{x_2}^b dx_1 a(x_1)Z^{2*}(x_1,\omega)\end{aligned}\tag{3.8a}$$

for even m integers. Similarly, for odd m integers, we obtain $R^{(m)}(x,\omega)=0$ and

$$\begin{aligned}L^{(m)}(x,\omega)=(-1)^{(m+1)/2}R^{(0)}(x,\omega)Z^2(x,\omega)\\ \times\int_x^b dx_m a(x_m)Z^{2*}(x_m,\omega)\\ \times\int_0^{x_m} dx_{m-1}a(x_{m-1})Z^2(x_{m-1},\omega)\times\cdots\\ \times\int_0^{x_3} dx_2 a(x_2)Z^2(x_2,\omega)\\ \times\int_{x_2}^b dx_1 a(x_1)Z^{2*}(x_1,\omega).\end{aligned}\tag{3.8b}$$

It follows that the full solution to the system (3.1) is given by

$$R(x,\omega)=R^0(x,\omega)+\sum_{\text{even},m=2}^{\infty}R^{(m)}(x,\omega),$$

$$L(x,\omega)=\sum_{\text{odd},m=1}^{\infty}L^{(m)}(x,\omega).\tag{3.9}$$

IV. ILLUSTRATION OF THE FREQUENCY DOMAIN TECHNIQUE FOR A SERIES OF DIELECTRIC SLABS

Let us first consider the simple case of a $\hat{y}$-polarized field $f(\omega)$ incident on a single dielectric slab of thickness b and dielectric constant $\varepsilon_0 n^2$. The index of refraction n can be considered complex if the medium is absorbing or amplifying [13,14]. The slab has sharp vacuum-medium interfaces at $x=0$ and b. Then the velocity $v(x)=c[1-(1-1/n)\theta(x)+(1-1/n)\theta(x-b)]$, where $c=(\mu\varepsilon_0)^{-1/2}$ is the velocity of the incident light in vacuum and the usual unit step function is defined as $\theta(x)=0$ for $x<0$, $\theta(x)=1/2$ for $x=0$, and $\theta(x)=1$ for $x>0$. Using the above expression for $v(x)$ we get

$$\begin{aligned}Z(x,\omega)=\exp\bigg[-\frac{i\omega}{c}\{x+x(n-1)\theta(x)\\ -(x-b)(n-1)\theta(x-b)\}\bigg],\end{aligned}\tag{4.1a}$$

$$a(x)=\frac{1}{2}\left[\frac{(1-1/n)\{\delta(x)-\delta(x-b)\}}{1-(1-1/n)\{\theta(x)-\theta(x-b)\}}\right].\tag{4.1b}$$

It can be noted from the definition (2.3) that, unlike the tangential and normal components of electric and magnetic fields, $L(x,\omega)$ and $R(x,\omega)$ are not continuous at the boundary, if $\varepsilon(x)$ is discontinuous. Integrating Eq. (3.8b) from $x=0$ to $x=b$, the resulting expression for the left going wave is

$$\begin{aligned}L^{(1)}(0,\omega)&=-\int_0^b dx\, a(x)Z^{2*}(x,\omega)f(\omega)\\ &=\gamma\left[1-\exp\left(\frac{2i\omega}{c}nb\right)\right]f(\omega),\end{aligned}\tag{4.2}$$

where $\gamma\equiv[(1-n)/(1+n)]$. A careful examination shows that expression (4.2) is the sum of two waves reflected at the interface at $x=0$ and $x=b$, respectively. Similarly integrating Eq. (3.8a) with the limits $x=0$ and $x=b$, we obtain

$$\begin{aligned}R^{(2)}(b,\omega)&=-\gamma L^{(1)}(0,\omega)\exp\left(\frac{i\omega}{c}nb\right)\\ &=-\gamma^2\exp\left(\frac{i\omega}{c}nb\right)\left[1-\exp\left(\frac{2i\omega}{c}nb\right)\right]f(\omega),\end{aligned}\tag{4.3}$$

where we have used the boundary condition $L^{(1)}(b,\omega)=0$. The usual transmission and reflection amplitudes used in the transfer matrix theory [1,6] are related to our scattered fields via $r(\omega)\equiv L(0,\omega)/f(\omega)$ and $t(\omega)\equiv R(b,\omega)\exp[-(i\omega/c)b]/f(\omega)$. This phase factor is necessary to reflect the fact that in the transfer matrix theory $t(\omega)=1$, whereas $R(b,\omega)=\exp[(i\omega/c)b]f(\omega)$ for vacuum $(n=1)$. Summing up all the iterative terms for $R(b,\omega)$ and $L(0,\omega)$ we obtain the series

$$\begin{aligned}r(\omega)&=\gamma\left\{1-\exp\left(\frac{2i\omega}{c}nb\right)\right\}\left[1+\gamma^2\exp\left(\frac{2i\omega}{c}nb\right)\right.\\ &\quad\left.+\gamma^4\exp\left(\frac{4i\omega}{c}nb\right)+\cdots\right]\\ &=\frac{\gamma\left[1-\exp\left(\frac{2i\omega}{c}nb\right)\right]}{\left[1-\gamma^2\exp\left(\frac{2i\omega}{c}nb\right)\right]},\end{aligned}\tag{4.4a}$$

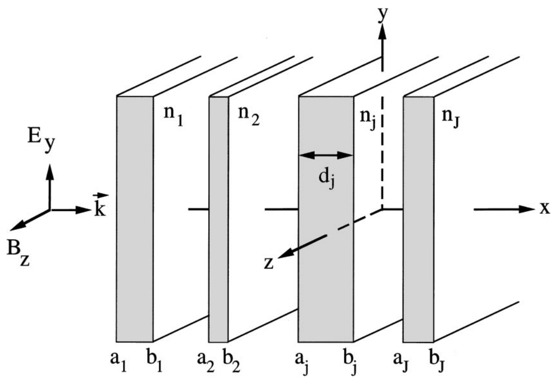

FIG. 1. Sketch of a layered medium with slabs arranged in a disordered fashion.

$$t(\omega)=\left[1-\gamma^2\left\{1-\exp\left(\frac{2i\omega}{c}nb\right)\right\}-\gamma^4\exp\left(\frac{2i\omega}{c}nb\right)\right.$$
$$\left.\times\left\{1-\exp\left(\frac{2i\omega}{c}nb\right)\right]-\cdots\right]\exp\left(\frac{i\omega}{c}(n-1)b\right)$$
$$=\frac{(1-\gamma^2)}{\left[1-\gamma^2\exp\left(\frac{2i\omega}{c}nb\right)\right]}\exp\left(\frac{i\omega}{c}(n-1)b\right). \quad (4.4b)$$

The factors $(1-\gamma^2)\exp[(i\omega/c)nb]$ in the expression (4.4b) are the product of the fraction of the wave transmitted at both interfaces $x=0$ and $x=b$. The factor $(1-\gamma^2)\times\gamma^2\exp[(i\omega/c)nb]\exp[(2i\omega/c)nb]$ represents a wave transmitted after two reflections, first backwards at $x=b$, then forward at $x=0$. Continuing similarly, the higher order contribution can be explained.

We next generalize this approach to a medium containing J slabs, arranged randomly along the x axis, with width $d_j\equiv b_j-a_j$, and dielectric constant $\varepsilon_0 n_j^2$. This sequence of dielectric slabs is shown in Fig. 1. The index of refraction n_j, the width of each layer d_j, and the spacing between the centers of the slabs Δx_j were chosen randomly with a uniform distribution in the range $1.3\leqslant n_j\leqslant 1.5$, $0.2\leqslant d_j/\overline{\Delta x}\leqslant 0.4$, and $0.5\leqslant \Delta x_j/\overline{\Delta x}\leqslant 1.5$, respectively. Here $\overline{\Delta x}$ denotes the average spacing between two adjacent slabs. In the following, all lengths become unitless and are expressed in terms of the scale length $\overline{\Delta x}$. If the incoming light is perpendicular to the surface of the slabs, the corresponding Maxwell equations can be reduced to the set of one-dimensional equations introduced in Eqs. (2.2). This illustrates nicely that the one-dimensionality is not merely a mathematical simplification but shows that the property $\varepsilon=\varepsilon(x)$ can be easily realized experimentally. Similar systems have been used in previous studies in connection with random lasers [13–15].

We can evaluate the relevant integrals leading to

$$Z(x,\omega)=\exp\left[-\frac{i\omega}{c}\left\{x+\sum_j^J (x-a_j)(n_j-1)\,\theta(x-a_j)\right.\right.$$
$$\left.\left.-\sum_j^J (x-b_j)(n_j-1)\,\theta(x-b_j)\right\}\right] \quad (4.5)$$

and

$$a(x)=\frac{\sum_j^J (1-1/n_j)\{\delta(x-a_j)-\delta(x-b_j)\}}{2\left[1-\sum_j^J (1-1/n_j)\{\theta(x-a_j)-\theta(x-b_j)\}\right]}. \quad (4.6)$$

Using Eqs. (3.7), the mth-order solution for the right- and left-moving wave can be obtained,

$$R^{(m)}(x,\omega)=\frac{Z^*(x,\omega)}{\beta(x)}\sum_{j=1}^{k-1}\gamma_j\{\beta(b_j)Z(b_j,\omega)L^{(m-1)}(b_j,\omega)$$
$$-\beta(a_j)Z(a_j,\omega)L^{(m-1)}(a_j,\omega)\}, \quad (4.7a)$$

$$L^{(m-1)}(x,\omega)=\frac{Z(x,\omega)}{\beta(x)}\sum_{j=k}^{J}\gamma_j\{\beta(a_j)Z^*(a_j,\omega)$$
$$\times R^{(m-2)}(a_j,\omega)-\beta(b_j)Z^*(b_j,\omega)$$
$$\times R^{(m-2)}(b_j,\omega)\}, \quad (4.7b)$$

where $k=1,\ldots,J$ and $b_{k-1}\leqslant x\leqslant a_k$ denotes the region between the $(k-1)$th and kth slab. The integrals in Eqs. (3.7) were reduced to a summation over the scatterers. The phase factors reduce to

$$Z(a_k,\omega)=\exp\left[-\frac{i\omega}{c}\left\{a_k+\sum_{j=1}^{k-1}(n_j-1)d_j\right\}\right]$$

and

$$Z(b_k,\omega)=\exp\left[-\frac{i\omega}{c}\left\{b_k+\sum_{j=1}^{k}(n_j-1)d_j\right\}\right].$$

In deriving Eq. (4.7) we have used the notation $\gamma_j\equiv(1-n_j)/(1+n_j)$. If we define the mth-order reflection and transmission amplitudes as $\mathbf{r}^{(m)}(x,\omega)\equiv L^{(m)}(x,\omega)\beta(x)/f(\omega)$ and $\mathbf{t}^{(m)}(x,\omega)\equiv R^{(m)}\times(x,\omega)\beta(x)/f(\omega)$, the following recurrence relations can be obtained:

$$\mathbf{r}^{(m-1)}(a_k,\omega)/Z(a_k,\omega)=\gamma_k\{\mathbf{t}^{(m-2)}(a_k,\omega)Z^*(a_k,\omega)$$
$$-\mathbf{t}^{(m-2)}(b_k,\omega)Z^*(b_k,\omega)\}$$
$$+\mathbf{r}^{(m-1)}(a_{k+1},\omega)/Z(a_{k+1},\omega), \quad (4.8a)$$

$$\mathbf{r}^{(m-1)}(b_k,\omega)/Z(b_k,\omega)=\gamma_{k+1}\{\mathbf{t}^{(m-2)}(a_{k+1},\omega)$$
$$\times Z^*(a_{k+1},\omega)$$
$$-\mathbf{t}^{(m-2)}(b_{k+1},\omega)Z^*(b_{k+1},\omega)\}$$
$$+\mathbf{r}^{(m-1)}(b_{k+1},\omega)/Z(b_{k+1},\omega), \quad (4.8b)$$

$$\mathbf{t}^{(m)}(a_k,\omega)/Z^*(a_k,\omega)=\gamma_{k-1}\{r^{(m-1)}(b_{k-1},\omega)Z(b_{k-1},\omega)-\mathbf{r}^{(m-1)}(a_{k-1},\omega)Z(a_{k-1},\omega)\}+\mathbf{t}^{(m)}(a_{k-1},\omega)/Z^*(a_{k-1},\omega), \tag{4.8c}$$

$$\mathbf{t}^{(m)}(b_k,\omega)/Z^*(b_k,\omega)=\gamma_k\{r^{(m-1)}(b_k,\omega)Z(b_k,\omega)-r^{(m-1)}(a_k,\omega)Z(a_k,\omega)\}+t^{(m)}(b_{k-1},\omega)/Z^*(b_{k-1},\omega). \tag{4.8d}$$

The boundary conditions can be rewritten as $\mathbf{r}^{(m)}(b_j,\omega)=0$ for all $m,\mathbf{t}^{(m)}(0,\omega)=0$ for $m\geqslant 2$ and $t^{(0)}(x,\omega)=Z^*(x,\omega)$. The system (4.8) can be solved for even m. The reflection and transmission coefficients are defined again through $r(\omega)\equiv L(0,\omega)/f(\omega)$ and $t(\omega)\equiv R(b,\omega)\exp[-(i\omega/c)b]/f(\omega)$, which take the form

$$r(\omega)=\sum_{\text{even},m=2}^{\infty} r^{(m-1)}(0,\omega), \tag{4.9a}$$

$$t(\omega)=\exp\left(-\frac{i\omega}{c}b\right)\left(Z^*(b,\omega)+\sum_{\text{even},m=2}^{\infty} t^{(m)}(b,\omega)\right). \tag{4.9b}$$

We show now numerically, for the medium with randomly arranged dielectric layers, a comparison between the exact solution $T\equiv|t(\omega)|^2$ and the iterative solution. In the top of Fig. 2, we have graphed the total transmission coefficient for the medium with $J=100$ layers as a function of the wavelength $\lambda=2\pi c/\omega$. The exact transmission coefficient is obtained using the transfer matrix theory [16,17].

In the range from $0<\lambda<8$ the transmission is characterized by rapid oscillations, a very small change in wavelength can change the medium from nearly transparent $(T\approx 0)$ to almost opaque $(T\approx 1)$. For larger wavelengths the transmission profile is less oscillatory and approaches $T=1$ in the limit of large wavelengths. We should note that this curve is for wavelength independent indices of refraction and is entirely due to the scattering at the interfaces. For a medium whose index of refraction varies directly with the wavelength of the incident light, the transmission curve would be modified. In the inset we have amplified the small wavelength window $10<\lambda<12$. We will use this range later in the discussion of the time-dependent pulses. The exact transmission is compared with the prediction of our iterative solutions derived above for $m=6$, 10, and 20. While the terms up to the sixth order give only a qualitative agreement, and for the range $10.3<\lambda<10.5$, T exceeds even the physical limit of $T=1$, the 20th-order iteration is practically indistinguishable from the exact curve and we have full convergence.

To show the behavior of the iterated transmission coefficients for the entire wavelength region we have displayed in the bottom figure the relative error $|T^{(m)}(\lambda)-T(\lambda)|/T(\lambda)$, where $T^{(m)}(\lambda)=|t(\lambda)|^2$ is the transmission coefficient obtained for an iteration up to order m in Eq. (4.9b). As the transmission coefficient gets larger for longer wavelength, the error of the iterative solutions decreases. This is expected because as T approaches unity, less and less higher-ordering scattering events become important.

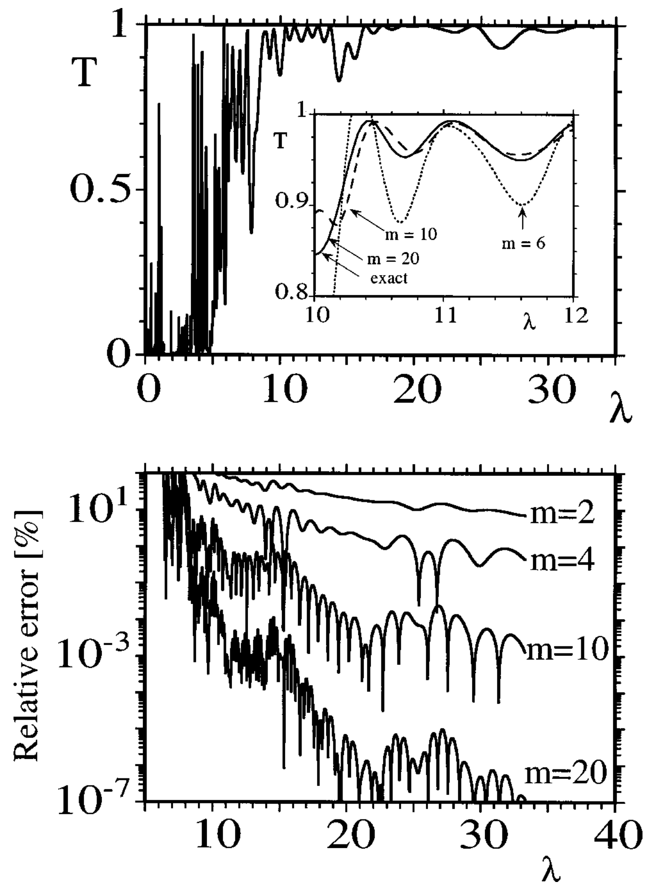

FIG. 2. (Top) The net transmission coefficient T as a function of wavelength λ from a medium containing $J=100$ slabs. The refractive index, the location, and the width of the slabs were randomly assigned from a uniform distribution in the range $1.3\leqslant n_j\leqslant 1.5$, $0.5\leqslant x_j/\overline{\Delta x}\leqslant 1.5$, and $0.2\leqslant d_j/\overline{\Delta x}\leqslant 0.4$, respectively. The inset shows the exact transmission coefficient compared to the transmission coefficients when evaluated up to various orders of iteration. (bottom) The relative error as a function of wavelength λ for the same medium as in the top figure. In all graphs, the wavelength is measured in units of $\overline{\Delta x}$, the average distance between two slabs.

V. THE ITERATIVE SOLUTIONS IN THE TIME DOMAIN

In order to study our system in its time dependence, let us make the following change of variables to eliminate the self-coupling terms in the system (2.4):

$$\tilde{R}(x,t)\equiv\beta(x)R(x,t),\quad \tilde{L}(x,t)\equiv\beta(x)L(x,t). \tag{5.1}$$

Upon substitution of Eqs. (5.1) into the system (2.4), we obtain the following equations:

$$\left(v(x)\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)\tilde{R}(x,t)=w(x)\tilde{L}(x,t), \tag{5.2a}$$

$$\left(v(x)\frac{\partial}{\partial x}-\frac{\partial}{\partial t}\right)\tilde{L}(x,t)=w(x)\tilde{R}(x,t), \tag{5.2b}$$

where the coupling strength is proportional to the gradient of the velocity $w(x)\equiv -\frac{1}{2}dv(x)/dx$. We can construct solutions that involve increasing orders of the $w(x)$. The zeroth-order iterative solutions are defined by

$$\left(v(x)\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)\tilde{R}^{(0)}(x,t)=0, \tag{5.3a}$$

$$\left(v(x)\frac{\partial}{\partial x}-\frac{\partial}{\partial t}\right)\tilde{L}^{(0)}(x,t)=0 \tag{5.3b}$$

subject to the initial conditions $R(x,t=0)\equiv f(x)$ and $L(x,t=0)\equiv 0$, i.e., initially there is no left going wave. We obtain $\tilde{R}^{(0)}(x,t)=f[T^{-1}(T(x)-t)]\beta[T^{-1}(T(x)-t)]$ and $\tilde{L}^{(0)}(x,t)=0$, where $T(x)\equiv\int_0^x dx'/v(x')$ is the time a pulse of velocity $v(x)$ will take to travel from $x=0$ to x without scattering. As $v(x)$ $[\equiv(\mu\varepsilon(x))^{-1/2}]$ is positive the inverse function $T^{-1}(y)$ exists such that $T^{-1}(T(x))=x$. We can generate recursively the mth-order solutions from the $(m-1)$th-order solutions by solving the equations

$$\left(v(x)\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)\tilde{R}^{(m)}(x,t)=w(x)\tilde{L}^{(m-1)}(x,t), \tag{5.4a}$$

$$\left(v(x)\frac{\partial}{\partial x}-\frac{\partial}{\partial t}\right)\tilde{L}^{(m)}(x,t)=w(x)\tilde{R}^{(m-1)}(x,t) \tag{5.4b}$$

subject to the initial conditions $\tilde{R}^{(m)}(x,t=0)=0$ and $\tilde{L}^{(m)}(x,t=0)=0$ for $m\geqslant 1$. This yields the recursive expressions

$$\tilde{R}^{(m)}(x,t)=\int_0^t dt' w[T^{-1}(t'-t+T(x))]\times\tilde{L}^{(m-1)}\times[T^{-1}(t'-t+T(x)),t'], \tag{5.5a}$$

$$\tilde{L}^{(m)}(x,t)=-\int_0^t dt' w[T^{-1}(t-t'+T(x))]\times\tilde{R}^{(m-1)}\times[T^{-1}(t-t'+T(x)),t']. \tag{5.5b}$$

Using Eqs. (5.1), we obtain $R^{(0)}(x,t)=f[T^{-1}(T(x)-t)]\beta[T^{-1}(T(x)-t)]/\beta(x)$ and $L^{(0)}(x,t)=0$. We should note that this complicated expression simplifies significantly when evaluated at positions x that are outside the medium $(x\geqslant b)$. In this case the minimum arrival time t must be larger than $T(x)$ which is the earliest time the pulse could arrive at location x without scattering. For a negative argument, $T(x)-t$, the inverse function is simply $T^{-1}(y)=cy$, associated with the propagation through vacuum before the pulse enters the medium at $x=0$. In other words, the function $T^{-1}(T(x)-t)$ reduces to the form $c[T(x)-t]$. This can be further simplified to $c(T(x)-t)=x-ct+x_d$, where the delay is defined as $x_d=cT(b)-b$. Similarly, we obtain $\beta(x)=1$ for $x\geqslant b$ and $\beta[T^{-1}(T(x)-t)]=1$ such that the transmitted zeroth-order pulse is $R^{(0)}(x,t)=f(x-ct+x_d)$, identical to a shifted initial pulse. For $m>0$, we obtain

$$R^{(m)}(x,t)=\frac{1}{\beta(x)}\int_0^t dt' w[T^{-1}(t'-t+T(x))]\beta[T^{-1}(t'-t+T(x))]L^{(m-1)}[T^{-1}(t'-t+T(x)),t'], \tag{5.6a}$$

$$L^{m}(x,t)=-\frac{1}{\beta(x)}\int_0^t dt' w[T^{-1}(t-t'+T(x))]\beta[T^{-1}(t-t'+T(x))]R^{(m-1)}[T^{-1}(t-t'+T(x)),t']. \tag{5.6b}$$

Note that $R^{(m)}(x,t)=0$ for odd integer m and $L^{(m)}(x,t)=0$ for even integer m because $L^{(0)}(x,t)=0$.

We now show how these complicated expressions become simpler if the initial right-going intensity is a short pulse, i.e., $R(x,t=0)=\delta(x)$. This case will also help us to interpret the time-dependent solution. In this case $R^{(0)}(x,t)=\delta[T^{-1}(T(x)-t)]\beta[T^{-1}(T(x)-t)]/\beta(x)$ and $L^{(0)}(x,t)=0$. By direct computation using Eqs. (5.6), we can simplify the first-order reflected light to

$$\begin{aligned}L^{(1)}(x,t)&=-\frac{1}{\beta(x)}\int_0^t dt' w[T^{-1}(t-t'+T(x))]\beta[T^{-1}(T(x)+t-2t')]\delta[T^{-1}(T(x)+t-2t')]\\&=-\frac{1}{\beta(x)}\int_{T^{-1}(T(x)-t)}^{T^{-1}(T(x)-t)}\frac{dy}{2v(y)}w[T^{-1}(t/2+T(x)/2+T(y)/2)]\beta(y)\,\delta(y)\\&=-\frac{1}{2c\beta(x)}w[T^{-1}(t/2+T(x)/2)],\quad t>T(x),\end{aligned} \tag{5.7}$$

where we write $v(0)=c$, the velocity of light in vacuum. For $t<T(x)$ the pulse does not have enough time to reach x and we get $L^{(1)}(x,t)=0$. Similarly, solving Eqs. (5.6) for $R^{(2)}(x,t)$ yields

$$R^{(2)}(x,t)=-\frac{1}{2c\beta(x)}\int_0^t dt' w[T^{-1}(t'-t+T(x))]\times w[T^{-1}(t'-t/2+T(x)/2)]. \tag{5.8}$$

At this point we can give an interpretation of these formulas. $R^{(2)}(x,t)$ is the total field at x and t of all the right-going fields that have scattered exactly two times. We introduce x_1 and x_2 as the positions of the first and second scattering events, respectively, with associated times t_1 and t_2. Thus, an initial right-going pulse travels from 0 to x_1 in time $t_1=T(x_1)$, scatters at x_1, travels left from x_1 to x_2 in time $T(x_1)-T(x_2)$, scatters at x_2, and then travels rightward from x_2 to x in time $T(x)-T(x_2)$. It should be borne in mind that $T(x_2)$ is not the same as t_2 because the latter

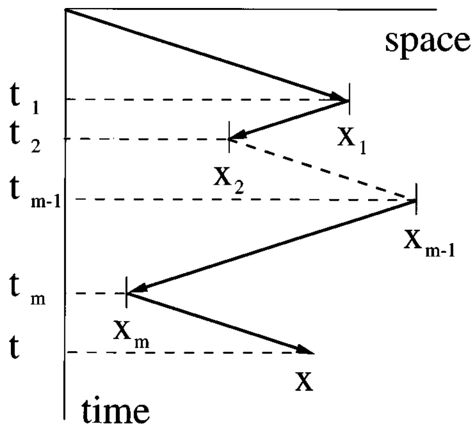

FIG. 3. Sketch of a typical scattering path on which the iterative approach is based.

represents the time the pulse will reach x_2 after multiple scattering, while $T(x_2)$ denotes time traveled along a straight line to reach from $x=0$ to $x=x_2$. In order for this pulse to arrive at x at time t, we have $T(x_1)+[T(x_1)-T(x_2)]+[T(x)-T(x_2)]=t$. Similarly, in order for this pulse to arrive at x_2 at time t_2, we must have $T(x_1)+[T(x_1)-T(x_2)]=t_2$. This labeling scheme is depicted in Fig. 3. Solving for x_1 and x_2 in terms of x, t, and t_2, we obtain $x_1=T^{-1}(T(x)/2-t/2+t_2)$ and $x_2=T^{-1}(T(x)-t+t_2)$. If we replace the integration variable t' by t_2 in the formula for $R^{(2)}(x,t)$ in Eq. (5.8), we obtain

$$R^{(2)}(x,t)=-\frac{1}{2c\beta(x)}\int_0^t dt_2 w(x_2)w(x_1). \tag{5.9}$$

Thus, we can now interpret $w(x)$ as the strength of direction reversal at position x and the integral corresponds to the sum of all the amplitudes of those fields that have scattered exactly twice before arriving at position x and time t.

We now apply this insight to the general solutions in Eqs. (5.6). For a fixed m, x_i is the position of the ith scattering event along the path of a pulse that scatters exactly m times before arriving at position x at time t. t_i denotes the time that the pulse arrives at x_i. Then as above we can change variables that yields, for m even ($m\geqslant 2$), $L^{(m)}(x,t)=0$, and

$$R^{(m)}(x,t)=(-1)^{m/2}\frac{1}{2c\beta(x)}\int_0^t dt_m\int_0^{t_m}dt_{m-1}\cdots \times\int_0^{t_3}dt_2 w(x_m)w(x_{m-1})\cdots w(x_1). \tag{5.10}$$

Similarly, we have, for odd integer m, $R^{(m)}(x,t)=0$ and

$$L^{(m)}(x,t)=(-1)^{(m+1)/2}\frac{1}{2c\beta(x)}\int_0^t dt_m\int_0^{t_m}dt_{m-1}\cdots \times\int_0^{t_3}dt_2 w(x_m)w(x_{m-1})\cdots w(x_1). \tag{5.11}$$

Thus, the system (2.4) with the initial conditions $R(x,t=0)=\delta(x)$ and $L(x,t=0)=0$ has the full solution

$$R(x,t)=\delta[T^{-1}(T(x-t))]\beta[T^{-1}(T(x)-t)]/\beta(x)+\sum_{\mathrm{even},m=2}^{\infty}R^{(m)}(x,t),$$

$$L(x,t)=\sum_{\mathrm{odd},m=1}^{\infty}L^{(m)}(x,t). \tag{5.12}$$

These expressions are closely related to the iterative solutions obtained in the frequency domain.

VI. NUMERICAL ILLUSTRATION OF THE ITERATIVE TECHNIQUE

In this section, we show the implementation of the above time domain iterative technique to demonstrate the dynamics of a pulse with Gaussian profile. We consider again J slabs of thickness d_j and refractive indices n_j ($j=1,...,J$). The velocity function for this medium is $v(x)=c[1-\Sigma_{j=1}^{J}(1-1/n_j)(\theta(x-a_j)-\theta(x-b_j))]$ and

$$T(x)=\frac{1}{c}\left[x+\sum_{j=1}^{J}(n_j-1)\{(x-a_j)\theta(x-a_j)-(x-b_j)\theta(x-b_j)\}\right]. \tag{6.1}$$

For this medium, $T(x)$ is not a continuous function. The direction reversal strength is obtained as

$$w(x)=\frac{c}{2}\sum_{j=1}^{J}\frac{n_j-1}{n_j}\{\delta(x-a_j)-\delta(x-b_j)\}. \tag{6.2}$$

Using Eq. (5.5) for and $m=2,4,6,...$, we get

$$\tilde{R}^{(m)}(x,t)=\int_{T^{-1}(T(x)-t)}^{x}dy\ a(y)\tilde{L}^{(m-1)}(y,t+T(y)-T(x)), \tag{6.3a}$$

$$\tilde{L}^{(m-1)}(x,t)=-\int_x^{T^{-1}(T(x)+t)}dy\ a(y)\tilde{R}^{(m-2)}(y,t+T(x)-T(y)). \tag{6.3b}$$

As noted previously, if the region between the $(k-1)$th and kth slab is denoted by x_k, then it is straightforward to show that for $t>0$, $\tilde{R}^{(m)}(x_I,t)=\tilde{L}^{(m-1)}(x_{J+1},t)=0$ and the recursion relation of the following type can be obtained:

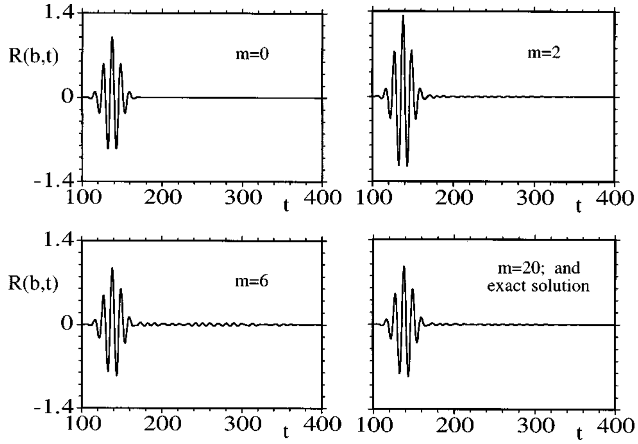

FIG. 4. The field $R(b,t)$ as a function of time, transmitted from a medium containing $J=100$ slabs. The disorder in the refractive index, location, and width of the slabs are the same as in Fig. 2. The figures show the transmitted pulse computed up to various orders of iteration and the exact solution. For all the plots, $b=100\overline{\Delta x}$ and time is measured in units of $\overline{\Delta x}/c$.

$$\begin{aligned}\tilde{R}^{(m)}(x_{k+1},t)=\gamma_k\{&\tilde{L}^{(m-1)}(b_k,t+T(b_k)-T(x_{k+1}))\\&-\tilde{L}^{(m-1)}(a_k,t+T(a_k)-T(x_{k+1}))\}\\&+\tilde{R}^{(m)}(x_k,t),\quad t>T(x_{k+1})-T(a_1)\end{aligned}\tag{6.4a}$$

$$\begin{aligned}\tilde{L}^{(m-1)}(x_k,t)=\gamma_k\{&\tilde{R}^{(m-2)}(a_k,t+T(x_k)-T(a_k))\\&-\tilde{R}^{(m-2)}(b_k,t+T(x_k)-T(b_k))\}\\&+\tilde{L}^{(m-1)}(x_{k+1},t),\quad t>T(b_N)-T(x_k').\end{aligned}\tag{6.4b}$$

Let us now give a numerical illustration of the iteration scheme for a finite pulse propagating through the same 100 random slabs as discussed in Fig. 2. In Fig. 4 we present the $m=0$th, 2nd, 6th, and 20th order solutions $R(x=100,t)$ generated from the set (5.6) for an incoming Gaussian pulse $R(x,t=0)=f(x)=[\exp(-(x-x_0))^2/2\sigma^2]\cos[(\omega/c)(x-x_0)]$ of width σ and centered at $x_0=-25$. The central frequency ω was chosen to be $\omega=0.5712c/\overline{\Delta x}$, corresponding to a wavelength $\lambda=11$. The spatial width $\sigma=10$ corresponds to a width in wavelength of $\Delta\lambda=2$. This range is precisely what has been displayed in the inset on the top of Fig. 2. The transmission coefficient varies between 85% and 99% in this wavelength regime.

The first figure shows the zeroth-order solution $R^{(0)}(x=100,t)$. This corresponds to the output pulse in the absence of scattering. In contrast to a pulse that has traveled through vacuum ($n=1$), however, $R^{(0)}$ arrives a little bit delayed, $R^{(0)}(x,t)=f(x-ct+x_d)$. This delay $x_d=13.5$ can be calculated from the medium as discussed above. For comparison, a pulse that had propagated through vacuum, $f(x-ct)$, would take its peak time at $ct=125$, whereas our pulse takes its largest value at time $ct=138.5$.

The fact that in our iteration scheme the zeroth-order solution agrees with a delayed pulse that has propagated through vacuum, and therefore has the same energy as the incoming pulse, has interesting consequences. For a highly scattering medium, the transmitted pulse is attenuated and quite different from the zeroth-order solution. Due to large amount of scatterings, the maximum of the exact pulse arrives at the right edge of the medium much later than the pulse $R^{(0)}$. In other words, the zeroth-order pulse predicts a too large amount of intensity at early arrival times and the higher-order iterates must correct this via destructive interference. This scenario is quite different from a similar iterative approach to the one-dimensional Boltzmann equation [18], in which all iterated solutions are probabilities and therefore positive and the higher-order terms cannot erase contributions from lower-order terms. In the Boltzmann case, the zeroth-order contribution corresponds to exactly that fraction of the transmitted light pulse that did not scatter, whereas in the present case of the Maxwell equations the total energy of the zeroth-order pulse is always larger than the total energy of the transmitted pulse.

The second-order iteration $R^{(2)}(x,t)$ can contain information about the trailing edge of the pulse. It should be noted that its tail for $150<ct<350$ is quite similar to that of the exact pulse. At the same time, the maximum amplitude ($=1.4$) is much larger than that of the exact pulse. The sixth-order solution $R^{(6)}(x,t)$ can reduce this amplitude by destructive interference, but the associated long time trail is too large. Finally, the 20th-order pulse is graphically indistinguishable from the exact pulse and the iterative scheme is converged. This convergence is expected from the inset in Fig. 2 for the individual wavelength components.

VII. SUMMARY AND OUTLOOK

In summary, we have derived an iterative method to solve the Maxwell equations for a one-dimensional model system with an arbitrary position-dependent dielectric constant. We have constructed solutions that are iterative in the scattering order, equivalent to the number of scattering events along the forward and backward directions.

A very difficult question concerns the generalization of this approach to two- or even three-dimensional systems. The feasibility of the method is based on the fact that the electric and magnetic field vectors can be rewritten in terms of a new left- and right-going field, respectively, which are coupled by the Maxwell equations. Following this procedure for a two-dimensional system would require the definition of a new vector field $M(\vec{r},\vec{v})$ as a function of $\vec{E}$ and $\vec{B}$ which is an explicit function of the propagation direction $\vec{v}$. The spatial and temporal evolutions of this field should be given by a Boltzmann-like generator of the form $\partial_t+\vec{v}\cdot\vec{\nabla}$ to permit the appropriate interpretation. However, we have not been able to construct such a field.

This work has been devoted to the derivation of this iterative scheme. Even though the main emphasis was on its numerical implementation, the lowest-order solutions provide fully analytical expressions that can be used for further investigations. One area of recent interest is to compare the solutions of the Boltzmann equation for one-dimensional medium with the exact ones from the Maxwell equations [16]. As a derivation of the Boltzmann equation from the Maxwell equation is still a challenge [19,20] and only small progress has been reported for special fields propagating through vacuum, our analytical solutions can be used to test various approximation schemes to better bridge the relationship between the Maxwell and Boltzmann description. We will report on these investigations elsewhere.

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